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Soliton Solutions of DNLS Equation Found by IST Anew and its Verification in Marchenko Formalism

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A general procedure is proposed to derive the multi-soliton solutions of DNLS equation with vanishing boundary value, and the two-soliton solutions of it is given as an example. Furthermore, the verification of multi-soliton solutions is done through Marchenko formalism.

KEY WORDS: nonlinear equation; soliton solution; Marchenko formalism.

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1. INTRODUCTION

Introducing a special spectral parameter κ (the reciprocal of the normal prameter λ), Kaup and Newell solved the derivative nonlinear Schrödinger equation (DNLS equation) with vanishing boundary condition through the inverse scattering transform(IST) method constructed on complex $\kappa-$plane (Kaup and Newell, 1978a,b). On the other hand, some other researchers, Wadati *et al.*, tried to do it on complex $\lambda-$plane (Wadati *et al.*, 1979), and all of them gave the same single soliton solution (Chen and Huang, 1989; Kaup and Newell, 1978c; Nakamura and CHen, 1980; Huang and Chen, 1990; Steudel, 2003; Kawata *et al.*, 1979). But the explicit expressions of multi-soliton solutions are not given so far. Recently, the perturbation theory of DNLS equation still attracts much attention (Kaup, 1990, 1991; Chen and Yang, 2002; Hao and Huang, 2005). As a result, the explicit expression of multi-soliton solutions of DNLS equation is required and the verification of it is also needed to be done.

As is well known, the single soliton solution can be verified finally by direct substitution into the nonlinear equations. But such a procedure is hard to be done for multi-soliton solutions and its verification should be treated seriously. In the

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case of nonlinear Schrödinger equation(NLS), its Jost solutions can be factorized as a product of Blaschiek's matrices in a recursive manner with the method of Darboux transformation matrix. As a result, if the single-soliton Jost solutions satisfy the Lax equations, the corresponding multi-soliton Jost solutions also do it in a recursive manner. And then the multi-soliton solutions certainly satisfy NLS equation by compatibility condition. On the other hand, it is uncertain that the Jost solutions derived from the equations of Cauchy integral satisfy the Lax equations because the pole values of it must be thought over. This problem does not exist in the Jost solutions derived from inverse scattering transform in Marchenko equation.

In the note, we obtain the exact multi-soliton solutions of DNLS equation through the standard inverse scattering transform method with introducing a factor λ^{-1} to ensure vanishing contribution of integral along the large circle in complex λ-plane as radius tending to infinite. And the explicit expression of two-soliton solutions is given as an example. In order to verify the multi-soliton solutions, the Jost solutions with same asymptotic behaviors are taken together to compose the 2×2 matrix $N(x, z; \lambda)$ that satisfies Marchenko equation. Then we can show the Jost solutions derived from inverse scattering transform in Marchenko formalism indeed satisfy both Lax equations. Finally, the multi-soliton solutions derived here satisfy the DNLS equation according to the compatibility condition actually.

2. DNLS EQUATION

The derivative nonlinear Schrödinger equation (DNLS) is given by

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0, \tag{1}$$

with vanishing boundary condition $u \to 0$, as $x \to \pm\infty$ and its Lax pairs are given by

$$\begin{aligned} L &= \lambda(-i\lambda\sigma_3 + U) \\ M &= -i2\lambda^4\sigma_3 + 2\lambda^3 U - i\lambda^2 U^2 \sigma_3 - \lambda(-U^3 + iU_x\sigma_3) \end{aligned} \tag{2}$$

and its Lax equations are

$$\partial_x \psi = L\psi, \quad \partial_t \psi = M\psi \tag{3}$$

From the first Lax equation, the free Jost solution $E(x, \lambda)$ is derived as

$$E(x, \lambda) = e^{-i\lambda^2 x \sigma_3} \quad \text{as} \quad x \longrightarrow \infty \tag{4}$$

The Jost solutions are proposed to

$$\Psi(x, \lambda) \equiv \left(\tilde{\psi}(x, \lambda), \psi(x, \lambda)\right) \to E(x, \lambda), \quad \text{as } x \to \infty \tag{5}$$

and

$$\Phi(x,\lambda) \equiv \big(\phi(x,\lambda), \tilde{\phi}(x,\lambda)\big) \to E(x,\lambda), \quad \text{as } x \to -\infty \tag{6}$$

Differing from the usual nonlinear Schrödinger (NLS) equation, the Jost solutions in the limit $|\lambda| \to \infty$ do not tend to zero. So it is necessary to introduce an additional factor λ^{-1} to construct the equations of inverse scattering transform by an integral of Cauchy's contour. This factor ensures vanishing contributions of the integral along the big circle in complex $\lambda-$plane as the radius tends to infinity.

3. PROPERTIES OF THE JOST SOLUTIONS

Since the first Lax equation of NLS is similar to that of DNLS, there are some same properties of the Jost solutions

$$\tilde{\psi}(x,\lambda) = -i\sigma_2\overline{\psi(x,\bar{\lambda})}, \quad \tilde{\phi}(x,\lambda) = i\sigma_1\overline{\phi(x,\lambda)} \tag{7}$$

$$\tilde{a}(\bar{\lambda}) = \overline{a(\lambda)}, \quad \tilde{b}(\lambda) = -\overline{b(\lambda)} \tag{8}$$

where $\psi(x,\lambda)$, $\phi(x,\lambda)$ and $a(\lambda)$ are analytic in the domain $\mathrm{Im}\lambda^2 > 0$, namely, in the I and III quadrants of complex $\lambda-$ plane. Moreover, since

$$L(x,-\lambda) = \sigma_3 L(x,\lambda)\sigma_3, \quad E(x,-\lambda) = \sigma_3 E(x,\lambda)\sigma_3 \tag{9}$$

we have the reduction transformation properties

$$\tilde{\psi}(x,-\lambda) = \sigma_3\tilde{\psi}(x,\lambda), \qquad \psi(x,-\lambda) = -\sigma_3\psi(x,\lambda) \tag{10}$$

$$\phi(x,-\lambda) = \sigma_3\phi(x,\lambda), \qquad \tilde{\phi}(x,-\lambda) = -\sigma_3\phi(x,\lambda) \tag{11}$$

$$\tilde{a}(-\lambda) = a(\lambda), \qquad \tilde{b}(-\lambda) = -b(\lambda) \tag{12}$$

From (7),

$$a(-\lambda_n) = a(\lambda_n). \tag{13}$$

if λ_n is a zero of $a(\lambda)$, $-\lambda_n$ is also a zero of $a(\lambda)$.

According to the Asymptotic behaviors in the limit $|\lambda| \to \infty$,we obtain

$$\bar{u} = -i \lim_{|\lambda|\to\infty} \frac{\lambda\tilde{\psi}_2(x,\lambda)}{\tilde{\psi}_1(x,\lambda)} \tag{14}$$

4. EQUATIONS OF INVERSE SCATTERING TRANSFORM

Defining

$$\Theta(x,\lambda) = \begin{cases} \frac{1}{a(\lambda)}\phi(x,\lambda), & \text{as } \lambda \text{ in I, III quadrants} \\ \tilde{\psi}(x,\lambda), & \text{as } \lambda \text{ in II, IV quadrants} \end{cases} \tag{15}$$

we have

$$\lambda^{-1}\{\Theta(x,\lambda)-E_{\cdot 1}(x,\lambda)\}e^{i\lambda^2 x}=\frac{1}{2\pi}\int_\Gamma d\lambda'\frac{1}{(\lambda'-\lambda)\lambda'}\{\Theta(x,\lambda')-E_{\cdot 1}(x,\lambda')\}e^{i\lambda'^2 x} \tag{16}$$

by introducing factor λ^{-1} to ensure vanishing contribution of the integral along the big arc as $|\lambda|\to\infty$. In the case of reflectionless we obtain

$$\tilde{\psi}(x,\lambda)=\begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda^2 x}+\lambda\{R(x,\lambda)\}e^{-i\lambda^2 x} \tag{17}$$

where

$$R(x,\lambda)=-\sum_n\frac{1}{\lambda-\lambda_n}\frac{1}{\lambda_n}c_n\psi(x,\lambda_n)e^{i\lambda_n^2 x},\qquad c_n=\frac{b_n}{\dot{a}(\lambda_n)} \tag{18}$$

Combining the terms of λ_n and $-\lambda_n$ by reduction transformation properties of (12) and (13), we have

$$\tilde{\psi}_1(x,\lambda)=e^{-i\lambda^2 x}+\lambda\sum_{n=1}^N\frac{2\lambda}{\lambda^2-\lambda_n^2}\frac{1}{\lambda_n}c_n\psi_1(x,\lambda_n)e^{i\lambda_n^2 x}e^{-i\lambda^2 x} \tag{19}$$

$$\tilde{\psi}_2(x,\lambda)=\lambda\sum_{n=1}^N\frac{2\lambda_n}{\lambda^2-\lambda_n^2}\frac{1}{\lambda_n}c_n\psi_2(x,\lambda_n)e^{i\lambda_n^2 x}e^{-i\lambda^2 x} \tag{20}$$

Substituting them into (14) yields

$$\bar{u}=\frac{V}{W} \tag{21}$$

where

$$W=1-\sum_{n=1}^N\frac{2}{\lambda_n}c_n\psi_1(x,\lambda_n)e^{i\lambda_n^2 x},\qquad V=-\sum_{n=1}^N 2c_n\psi_2(x,\lambda_n)e^{i\lambda_n^2 x} \tag{22}$$

5. N- SOLITON SOLUTIONS

Introducing some matrixes

$$g_n=\sqrt{2c_n}f_0(\lambda_n),\quad \Psi_{jn}=\sqrt{2c_n}\psi_j(\lambda_n) \tag{23}$$

$$B_{nm}=\overline{g}_n\frac{p_m}{\left(\overline{p}_n^2-p_m^2\right)}g_m,\quad B'_{nm}=\overline{g}_n\frac{\overline{p}_n^2}{p_m\left(\overline{p}_n^2-p_m^2\right)}g_m \tag{24}$$

$$p_n=i\lambda_n \tag{25}$$

where (23) and (25) are $1 \times N$ matrixes, (24) are $N \times M$ matrixes, and $f_0(\lambda) = e^{i\lambda^2 x}$.
Then (21) can be expressed with the form of matrix.

$$\overline{u}_2 = -2\frac{\Psi_2 g^{\mathrm{T}}}{1 - \Psi_1 P^{-1} g^{\mathrm{T}}} = -2\frac{(\det(I + R') - \det(I + R))\det(I + R'')}{\det(I + R)\det(I + R''')} \tag{26}$$

where

$$R = \overline{B}\,\overline{B}'^{\mathrm{T}}, \qquad R' = R + g^{\mathrm{T}} g, \tag{27}$$

$$R'' = BB'^{\mathrm{T}}, \quad R''' = R'' + P^{-1} B g^{\mathrm{T}} \overline{g} \tag{28}$$

and

$$\det(I + R) = \det(I + R'''), \quad \det(I + R) = \overline{\det(I + R'')} \tag{29}$$

With some algebra formulas, we can obtain the N-soliton solutions of DNLS equation. As an example, we give the exact expression of the two-soliton solutions:

$$\begin{aligned}\det(I + R) = {} & 1 + |f_1|^4 \frac{\lambda_1}{\overline{\lambda}_1}\left|\frac{\lambda_1^2 - \overline{\lambda}_2^2}{\lambda_1^2 - \lambda_2^2}\right|^2 + |f_2|^4 \frac{\lambda_2}{\overline{\lambda}_2}\left|\frac{\lambda_2^2 - \overline{\lambda}_1^2}{\lambda_2^2 - \lambda_1^2}\right|^2 \\ & + \left(f_2^2 \overline{f}_1^2 \frac{\lambda_2}{\overline{\lambda}_1} + f_1^2 \overline{f}_2^2 \frac{\lambda_1}{\overline{\lambda}_2}\right) \frac{(\lambda_1^2 - \overline{\lambda}_1^2)(\overline{\lambda}_2^2 - \lambda_2^2)}{(\lambda_1^2 - \lambda_2^2)(\overline{\lambda}_2^2 - \overline{\lambda}_1^2)} \\ & + |f_1|^4 |f_2|^4 \frac{\lambda_1 \lambda_2}{\overline{\lambda}_1\,\overline{\lambda}_2}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\det(I + R') - \det(I + R) = {} & -i\frac{\lambda_2^2 - \overline{\lambda}_2^2}{\lambda_2}\frac{\overline{\lambda}_1^2 \overline{\lambda}_2^2}{\lambda_1^2 \lambda_2^2} f_2^2 \left\{\frac{\lambda_2^2 - \overline{\lambda}_1^2}{\lambda_2^2 - \lambda_1^2} + \frac{\overline{\lambda}_2^2 - \lambda_1^2}{\overline{\lambda}_2^2 - \overline{\lambda}_1^2}|f_1|^4 \frac{\overline{\lambda}_1}{\lambda_1}\right\} \\ & + (\text{the other term of exchanging one with two})\end{aligned} \tag{31}$$

Substituting (30) and (31) into (26), we obtain the exact two-soliton solutions.

In Fig. 1, the three-dimensional figure of the propagation of two-soliton solutions is shown, where $\lambda_1 = \cos(0.9\pi/4) + i\,\sin(0.9\pi/4)$ and $\lambda_1 = \cos(1.1\pi/4) + i\,\sin(1.1\pi/4)$ .The two waves transmit respectively except for encounter that accords with the character of the propagation of soliton. When they encounter, the intermediate region between two peaks rapidly grows, which presents the procedure of nonlinear collision.

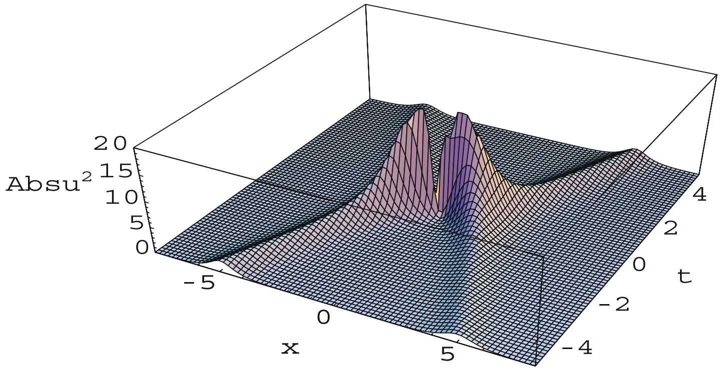

Fig. 1. 3D Plot of two-solition solutions under the vanishing condition.

6. MARCHENKO EQUATION FOR THE DNLS EQUATION

For the multi-soliton solutions, it is necessary to prove their correctness. We can not verify them with the method of direct substitution. Then we give the theoretic certification in Marchenko formulas. Introduce integral representation of Jost solutions

$$\Psi(x,\lambda) = E(x,\lambda) + \int_x dz\, N(x,z;\lambda)E(z,\lambda) \tag{32}$$

where $E(x,\lambda) = e^{-i\lambda^2 x\sigma_3}$. $N(x,z;\lambda)$ is a 2×2 matrix, and can be expressed as

$$N(x,z;\lambda) = \lambda^2 N^d(x,z) + \lambda N^{nd}(x,z) \tag{33}$$

where the indexes d and nd mean the diagonal and off-diagonal parts respectively. And by (7)

$$N_{11}(x,y) = \overline{N_{22}(x,y)}, \quad N_{21}(x,y) = -\overline{N_{12}(x,y)} \tag{34}$$

Based on the work of by Kaup and Newell (Kaup and Newell, 1978b), the Marchenko integral equations are

$$N_{12}(x,y) + \overline{f(x+y)} + \int_x^\infty dz\, N_{11}(x,z)\overline{f'(z+y)} = 0 \tag{35}$$

$$N_{11}(x,y) - \int_x^\infty dz\, N_{12}(x,z) f(z+y) = 0 \tag{36}$$

where

$$f(x+y) = i\sum_{n=1}^{N} \frac{1}{\lambda_n}\frac{b_n}{\dot{a}(\lambda_n)} e^{i\lambda_n^2(x+y)}, \quad f'(x+y) = i\sum_{n=1}^{N} \lambda_n \frac{b_n}{\dot{a}(\lambda_n)} e^{i\lambda_n^2(x+y)} \tag{37}$$

7. DEMONSTRATION OF THE FIRST ONE OF THE LAX EQUATION

Noticing $\overline{f(x+y)}_y = \overline{f(x+y)}_x = -i\overline{f'(x+y)}$, we give the derivative of (35) with respect to x and y respectively,

$$N_{12x}(x,y) - (i + N_{11}(x,x))\overline{f'(x+y)} + \int_x^{\infty} dz\, N_{11x}(x,z)\overline{f'(z+y)} = 0 \quad (38)$$

$$N_{12y}(x,y) - i\overline{f'(x+y)} + \int_x^{\infty} dz\, N_{11}(x,z)\overline{f'(z+y)}_y = 0 \quad (39)$$

and considering

$$\int_x^{\infty} dz\, N_{11}(x,z)\overline{f'(z+y)}_y = -N_{11}(x,x)\overline{f'(x+y)} - \int_x^{\infty} dz\, N_{11z}(x,z)\overline{f'(z+y)} \quad (40)$$

$$N_{22y}(x,y) = N_{21}(x,x)\overline{f(x+y)} + \int_x^{\infty} dz\, N_{21z}(x,z)\overline{f(z+y)} \quad (41)$$

we obtain

$$N_{12x}(x,y) - N_{12y}(x,y) - iu(x)N_{22y}(x,y) = 0 \quad (42)$$

So in the case of reflectionless, (42) can be written as the form of integral in a system of linear algebraic equation,

$$\begin{aligned} &N_{12x}(x,y) - N_{12y}(x,y) - iu(x)N_{22y}(x,y) \\ &\quad + \int_x^{\infty} dz\{N_{12x}(x,z) - N_{12z}(x,z) - iu(x)N_{22z}(x,z)\} = 0 \end{aligned} \quad (43)$$

then the first Lax equation is verified in the case of reflectionless.

8. DEMONSTRATION OF THE SECOND ONE OF THE LAX EQUATION

Taken account of the time-dependence, $E(x,\lambda)$ becomes

$$E(x,t,\lambda) = e^{-i\lambda^2(x+2\lambda^2 t)\sigma_3} \quad (44)$$

The integral representation is now

$$\Psi(x,t,\lambda) = E(x,t,\lambda) + \int_x dz(\lambda^2 N^d(x,z) + \lambda N^{nd}(x,z,t))E(z,t,\lambda) \quad (45)$$

The time derivative of (45) is

$$\begin{aligned} \partial_t \Psi(x,t,\lambda) &= -i2\lambda^4\sigma_3 E(x,t,\lambda) \\ &\quad + \lambda^2 \int_x dz\left\{N_t^d(x,z,t)E(z,t,\lambda) + N^d(x,z,t)2i\sigma_3 E''(z,t,\lambda)\right\} \\ &\quad + \lambda \int_x dz\left\{N_t^{nd}(x,z,t)E(z,t,\lambda) + N^{nd}(x,z,t)2i\sigma_3 E''(z,t,\lambda)\right\} \end{aligned} \quad (46)$$

and

$$\hat{M}\Psi(x,t,\lambda) = \hat{M}E(x,t,\lambda) + \hat{M}\int_x^\infty dz\, N(x,z,t;\lambda)E(z,t,\lambda) \tag{47}$$

After a series of integration by parts for (46) and (47),then comparing with the two sides of the equation, the terms of $\lambda \int_x dz\, E(z,\lambda)$ and $\lambda^2 \int_x dz\, E(z,\lambda)$ are

$$N_{11t}(x,z,t) + (|u|^2 u - iu_x)N_{21}(x,z) + 2iu(x)N_{21z}(x,z) - |u|^2 N_{11z}(x,z) = 0 \tag{48}$$

$$\begin{aligned} &N_{12t}(x,y,t) - 4iN_{12yy}(x,y,t) + 2u(x)N_{22yy}(x,y,t) \\ &- |u|^2 N_{12y}(x,y,t) + i(|u|^2 u(x) + iu(x)_x)N_{22y}(x,y,t) = 0 \end{aligned} \tag{49}$$

Considering the time-dependence, c_n is simply replaced by

$$c_n \to c_n(0)e^{4i\lambda_n^4 t} \tag{50}$$

then from (35) we can obtain

$$\Xi(x,y) + \int_x^\infty dz \Theta(x,z)\overline{f'(z+y)} = 0, \;\; \Theta(x,y) - \int_x^\infty dz\, \Xi(x,z) f(z+y) = 0 \tag{51}$$

where

$$\begin{aligned} \Xi(x,y) = {} & N_{12t}(x,y,t) - 4iN_{12yy}(x,y,t) + 2u(x)N_{22yy}(x,y,t) \\ & -|u|^2 N_{12y}(x,y,t) + i(|u|^2 u(x) + iu(x)_x)N_{22y}(x,y,t) \end{aligned} \tag{52}$$

$$\begin{aligned} \Theta(x,y) = {} & N_{11t}(x,z,t) + (|u|^2 u - iu_x)N_{21}(x,z) \\ & + 2iu(x)N_{21z}(x,z) - |u|^2 N_{11z}(x,z) \end{aligned} \tag{53}$$

In the case of reflectionless, $N(x,y)$ can be expressed as

$$N(x,y) = \begin{pmatrix} \sum_n N_{11}(x,\lambda_n)e^{i\lambda_n^2 y} & \sum_n N_{12}(x,\lambda_n)e^{-i\overline{\lambda}_n^2 y} \\ \sum_n N_{21}(x,\lambda_n)e^{i\lambda_n^2 y} & -\sum_n N_{22}(x,\lambda_n)e^{-i\overline{\lambda}_n^2 y} \end{pmatrix} \tag{54}$$

$$\Xi(x,y) = \sum_n \Xi(x,\lambda_n)e^{-i\overline{\lambda}_n^2 y}, \quad \Theta(x,y) = \sum_n \Theta(x,\lambda_n)e^{i\lambda_n^2 y} \tag{55}$$

Substituting (55) into (51), and integrating them, we can obtain

$$\sum_n \Theta(x,\lambda_n)e^{i\lambda_n^2 y} - \sum_{nm} \Xi(x,\lambda_m)\frac{1}{\lambda_n}\frac{c_n}{i\left(\lambda_n^2 - \overline{\lambda}_m^2\right)} e^{-i\left(\overline{\lambda}_m^2 - \lambda_n^2\right)x} e^{i\lambda_n^2 y} = 0 \tag{56}$$

and

$$\sum_n \Xi(x,\lambda_n)e^{-i\overline{\lambda}_n^2 y} + \sum_{nm} \Theta(x,\lambda_m)\overline{\lambda}_n \frac{\overline{c}_n}{-i\left(\overline{\lambda}_n^2 - \lambda_m^2\right)x} e^{-i\left(\overline{\lambda}_n^2-\lambda_m^2\right)x} e^{-i\overline{\lambda}_n^2 y} = 0 \tag{57}$$

Substituting (57) into (56), we can obtain

$$\sum_n \Theta(x,\lambda_n)e^{i\lambda_n^2 y} + \sum_{nm} \Theta(x,\lambda_n)A_{mn}\overline{B}_{mn}e^{i\lambda_n^2 y} = 0 \tag{58}$$

where

$$A_{mn} = \frac{1}{\lambda_n}\frac{c_n}{i\left(\lambda_n^2 - \overline{\lambda}_m^2\right)}, \qquad B_{mn} = \lambda_n \frac{c_n}{i\left(\lambda_n^2 - \overline{\lambda}_m^2\right)} \tag{59}$$

then (57) becomes

$$\sum_{m,n} \Theta(x,\lambda_n)e^{i\lambda_n^2 y}(\delta_{mn} + A\overline{B}) \neq 0 \tag{60}$$

So

$$\Theta(x,\lambda_n) = 0 \tag{61}$$

as the same procedure,we obtain

$$\Xi(x,\lambda_n) = 0 \tag{62}$$

The second Lax equation is proved.

9. CONCLUSION

In this note, a general method is provided to obtain the exact multi-soliton solutions of DNLS with vanishing boundary condition. At the same time, we demonstrate the Jost solutions obtained by inverse scattering transform in reflectionless case indeed satisfy the two Lax equations and the multi-soliton solutions obtained by the IST method satisfy the DNLS equation by the compatibility condition actually.

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